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Construction of the effective action in nonanticommutative supersymmetric field theories

O.D. Azorkina*[1], A.T. Banin†[2], I.L. Buchbinder‡[3]§, N.G. Pletnev¶[2]

[1] *Department of Theoretical Physics*
Tomsk State Pedagogical University
Tomsk 634041, Russia
[2] *Department of Theoretical Physics*
Institute of Mathematics, Novosibirsk,
630090, Russia
[3] *Department of Applied Mathematics and Theoretical Physics*
University of Cambridge, Centre for Mathematical Sciences
Wilberforce Road, Cambridge, CB3 0WA, UK

Abstract

We develop a general gauge invariant construction of the one-loop effective action for supersymmetric gauge field theories formulated in $\mathcal{N} = 1/2$ superspace. Using manifestly covariant techniques (the background superfield method and proper-time representations) adopted to the $\mathcal{N} = 1/2$ superspace we show how to define unambiguously the effective action of a matter multiplet (in fundamental and adjoint representations) and the vector multiplet coupled to a background $\mathcal{N} = 1/2$ gauge superfield. As an application of this construction we exactly calculate the low-energy one-loop effective action of matter multiplet and $SU(2)$ SYM theory on the Abelian background.

1 Introduction

Recently it was shown [1] that the low-energy limit of superstring theory in the self-dual graviphoton background field $F^{\alpha\beta}$ leads to a four-dimensional supersymmetric field theory formulated in the deformed $\mathcal{N} = 1$ superspace with fermionic coordinated satisfying the relation

$$\{\hat{\theta}^\alpha, \hat{\theta}^\beta\} = 2\alpha'^2 F^{\alpha\beta} = 2\mathcal{C}^{\alpha\beta}. \qquad (1)$$

*azorkina@tspu.edu.ru
†atb@math.nsc.ru
‡joseph@tspu.edu.ru, J.Buchbinder@damtp.cam.ac.uk
§On leave of absence from Department of Theoretical Physics, Tomsk State Pedagogical University, Tomsk 634041, Russia
¶pletnev@math.nsc.ru

The anti-commutation relations of the remaining $\mathcal{N}=1$ chiral superspace coordinates $y^m, \bar{\theta}^{\dot{\alpha}}$ are not modified. The field theories defined on such superspace can be formulated via ordinary superfield actions where the superfields product is defined via the star product [2]

$$f(x,\theta,\bar{\theta}) \star g(x,\theta,\bar{\theta}) = f \mathrm{e}^{\overleftarrow{Q_\alpha} \mathcal{C}^{\alpha\beta} \overrightarrow{Q_\beta}} g. \tag{2}$$

Here $Q_\alpha = i\frac{\partial}{\partial\theta^\alpha} + \frac{1}{2}\bar{\theta}^{\dot{\alpha}}\partial_{\alpha\dot{\alpha}}$ is the supersymmetry generator. Since the star-product contains explicitly the supersymmetry generator Q_α a half of supersymmetries is broken down. Therefore this superspace is called $\mathcal{N}=1/2$ superspace and the corresponding field theories are called $\mathcal{N}=1/2$ or nonanticommutative (NAC) supersymmetric theories. We emphasize that such a superspace deformation is possible only in the Euclidean space[1].

Renormalization of various $\mathcal{N}=1/2$ supersymmetric theories was discussed in Refs.[4], [5]. One-loop calculations of counterterms in supergauge models were carried out in component formalism [6] and in superspace [7]. It has been found [6], that at one loop, in the standard class of gauges, the non-gauge-invariant divergent terms are generated. However, there exists a non-linear redefinition of the gaugino fields which remove such terms and restores the gauge invariance. Moreover a modified version of the original pure $\mathcal{N}=1/2$ Lagrangian has been proposed in [6]. It has a form preserved under renormalization and the parameter of nonanticommutativity $\mathcal{C}^{\alpha\beta}$ is unrenormalized (at least at one loop). The one loop divergences for NAC $U(N)$ gauge theories with the matter in the adjoint representation have been studied in Ref. [7] using superfield background field method. It was found that the divergent non-gauge -invariant contributions are generated, however it was proved that sum of all one-loop divergences is gauge invariant without any redefinitions. We point out that practically all results in $\mathcal{N}=1/2$ gauge theories concerned only structure of divergences, the finite part of effective action has not been investigated.

The gauge invariance of the above results means that at the quantum level supergauge invariance is consistent with NAC geometry and hence there exists a procedure to construct the effective action in a form which preserves the manifestly gauge invariance and $\mathcal{N}=1/2$ supersymmetry. The aim of this paper is to formulate the procedure and study some of its applications. We consider $\mathcal{N}=1/2$ SYM theory coupled to a matter in fundamental and adjoint representations and use the superfield background field method (see e.g. [8]) together with superfield heat kernel method [9]. For evaluation of one-loop effective action we also use the special techniques developed in Refs. [10] adapting it to $\mathcal{N}=1/2$ superspace.

The paper is organize as follows. In the Section 2 we formulate the basic properties of nonanticommutative star-product and introduce the operators T^*_c and T^*_s which are very important for considering the NAC matter in the adjoint representation and pure NAC SYM theory. Section 3 is devoted to formulation of the models in superspace and superfield background field method. In the Section 4 we describe a calculation of the one-loop effective action for the matter in the fundamental representation coupled to an external SYM field and find the one-loop effective action on a covariantly constant on-shell background. Section 5 is devoted to discussion of the heat kernel techniques and the one-loop effective action for the NAC SYM theory. The obtained results are formulated

[1] In the case of extended supersymmetric theories, the various superspace deformations can be constructed in the sector of fermionic coordinates [3].

in the Summary. We do not discuss the details of calculations which are analogous to ones in conventional superfield theories (see e.g [8], [9], [10], [11], [12]) and pay attention only on the aspects essentially associated with $\star$-operation.

2 The properties of $\star$-product

The field theories on the $\mathcal{N}=1/2$ superspace can be conveniently formulated using a notion of symbols of the operators. For any operator function $\hat{f}$ depending on variables $\hat{\theta^\alpha}$ satisfying the relations (1) one defines the corresponding Weyl symbol $f(\theta)$ by the rule $\hat{f}=\int d^2\pi \mathrm{e}^{\pi\hat{\theta}}\tilde{f}(\pi)$ where $\tilde{f}(\pi)$ is the Fourier transform of the symbol $f(\theta)$: $\tilde{f}(\pi)=-\int d^2\theta \mathrm{e}^{-\pi\theta}f(\theta)$ (see some details in [13]). The delta function of the anticommuting variables θ is presented as $\delta(\theta-\theta')=\int d^2\pi \mathrm{e}^{\pi(\theta-\theta')}$. The set of operator functions forms a graded algebra. The in product of two operators $\hat{f},\hat{g}$ is associated with the star-product of the corresponding Weyl symbols $\hat{f}\cdot\hat{g}=\int d^2\pi \mathrm{e}^{\pi\hat{\theta}}\widetilde{(f\star g)}(\pi)$, where the star-product is defined by

$$f(\theta)\star g(\theta)=f(\theta)e^{-\frac{\overleftarrow{\partial}}{\partial\theta^\alpha}\mathcal{C}^{\alpha\beta}\frac{\overrightarrow{\partial}}{\partial\theta^\beta}}g(\theta)=f\cdot g-(-1)^{|f|}\frac{\partial f}{\partial\theta^\alpha}\mathcal{C}^{\alpha\beta}\frac{\partial g}{\partial\theta^\beta}-\frac{1}{2}\frac{\partial^2 f}{\partial\theta^2}\mathcal{C}^2\frac{\partial^2 g}{\partial\theta^2}. \qquad (3)$$

A star-product of exponential factors $\mathrm{e}^{\theta\pi}\star\mathrm{e}^{\theta\psi}=\mathrm{e}^{\theta\pi+\theta\psi-\pi_\alpha\mathcal{C}^{\alpha\beta}\psi_\beta}$ allows to write a star-product for two Weyl symbols in the form $f\star g=\int d^2\pi\, \mathrm{e}^{\pi\theta}f(\theta^\alpha+\mathcal{C}^{\alpha\beta}\pi_\beta)\tilde{g}(\pi)$, as well as the star-products of symbols and delta-function in the form

$$f_1\star\ldots\star f_n\star\delta(\theta-\theta')\star g_1\star\ldots\star g_m= \qquad (4)$$
$$\int d^2\pi\, \mathrm{e}^{(\theta-\theta')\pi}f_1(\theta+\mathcal{C}\pi)\star\ldots\star f_n(\theta+\mathcal{C}\pi)\star g_1(\theta-\mathcal{C}\pi)\star\ldots\star g_m(\theta-\mathcal{C}\pi)\ .$$

One can note that the change of variables $\theta\to\theta\pm\mathcal{C}\pi$ in the chain of the left (right) star-products of symbols and delta-functions does not affect on the exponent argument because of the property $\pi\mathcal{C}\pi=0$, however it simplifies arguments of f_i, for example:

$$f_1\star\ldots\star f_{n-1}\star f_n\star\delta(\theta-\theta')=\int d^2\pi\, \mathrm{e}^{(\theta-\theta')\pi}f_1(\theta)\star\ldots\star f_{n-1}(\theta)\star f_n(\theta)\ . \qquad (5)$$

These are the basic properties of the star-products which allows to adopt the rules of operation with superfields in the conventional superspace theory for the NAC superspace[2].

Using the star-product operation (3) we define the star-operators $T^\star_c$ and $T^\star_s$ by the rule

$$T^\star_c=\frac{1}{2}(\star+\star^{(-1)})=\cosh(\frac{\overleftarrow{\partial}}{\partial\theta^\alpha}\mathcal{C}^{\alpha\beta}\frac{\overrightarrow{\partial}}{\partial\theta^\beta}),\quad T^\star_s=\frac{1}{2}(\star-\star^{(-1)})=\sinh(-\frac{\overleftarrow{\partial}}{\partial\theta^\alpha}\mathcal{C}^{\alpha\beta}\frac{\overrightarrow{\partial}}{\partial\theta^\beta}) \qquad (6)$$

where $\star^{(-1)}$ means (3) with replacement $\mathcal{C}\to-\mathcal{C}$. There is the Leibniz rule for $T^\star_{c,s}$ products $\nabla_A(fT^\star g)=(\nabla_A f)T^\star g+fT^\star\nabla_A g$ and the Jacoby identity $fT^\star gT^\star h+hT^\star fT^\star g+gT^\star hT^\star f=0$ for integrand. It is easily to understand that any chain of $T^\star_s$-products is the total derivative

$$fT^\star_s g=(-1)^{|f|}\partial_\alpha(f\mathcal{C}^{\alpha\beta}\partial_\beta g). \qquad (7)$$

Further we will see that the operators T^*_c and T^*_s are very important for evaluation of effective action for fields in adjoint representation.

[2] See some analogous rules in noncommutative non-supersymmetric field theory in [14]

3 SYM theory coupled to the chiral matter in $\mathcal{N} = 1/2$ superspace and the background field method

The $\mathcal{N} = 1/2$ SYM theory in four dimensional superspace with explicitly broken supersymmetry in the antichiral sector can be defined on a $\mathcal{N} = 1$ superspace [2] as straightforward generalization of the standard construction (see e.g. [8], [9]) by introducing the NAC (but associative) star-product (2). The gauge transformations of (anti)chiral superfields in fundamental representation are given in terms of two independent chiral and antichiral parameter superfields $\Lambda, \bar{\Lambda}$ as follows

$$\Phi' = \mathrm{e}_\star^{i\Lambda} \star \Phi, \quad \bar{\Phi}' = \bar{\Phi} \star \mathrm{e}_\star^{-i\bar{\Lambda}}.$$

Gauge fields and field strengths together with their superpartners can be organized into superfields, which are expressed in terms of a scalar superfield potential V in the adjoint representation of the gauge group. The gauge transformations of V look like

$$\mathrm{e}_\star^{V'} = \mathrm{e}_\star^{i\bar{\Lambda}} \star \mathrm{e}_\star^{V} \star \mathrm{e}_\star^{-i\Lambda}.$$

Studying of component structure of supergauge theories is simplified with the help of Wess-Zumino (WZ) gauge. It was shown [2] that the commutative gauge transformation does not preserve the WZ gauge because of the properties of $\star$-product and one needs to perform an additional $\mathcal{C}$-dependent gauge transformation in order to recover the WZ gauge. Therefore the supersymmetry transformations of the component fields receive a deformation stipulated by the parameter of nonanticommutativity $\mathcal{C}^{\alpha\beta}$.

In the gauge chiral representation the constraints for the superspace covariant derivatives are solved by

$$\nabla_A = D_A - i\Gamma_A = (\mathrm{e}_\star^{-V} \star D_\alpha \mathrm{e}_\star^{V}, \bar{D}_{\dot{\alpha}}, -i\{\nabla_\alpha, \star\bar{\nabla}_{\dot{\alpha}}\}) \tag{8}$$

and the corresponding superfield strengths are given by the deformed algebra of the covariant derivatives

$$i\nabla_{\alpha\dot{\alpha}} = [\nabla_\alpha, \star\bar{\nabla}_{\dot{\alpha}}], \quad [\bar{\nabla}_{\dot{\alpha}}, \star\nabla_{\beta\dot{\beta}}] = \epsilon_{\dot{\alpha}\dot{\beta}} W_\beta, \quad [\nabla_\alpha, \star\nabla_{\beta\dot{\beta}}] = \epsilon_{\alpha\beta} \bar{W}_{\dot{\beta}} \tag{9}$$

$$[\nabla_{\alpha\dot{\alpha}}, \star\nabla_{\beta\dot{\beta}}] = -i(\epsilon_{\dot{\alpha}\dot{\beta}} f_{\alpha\beta} + \epsilon_{\alpha\beta} \bar{f}_{\dot{\alpha}\dot{\beta}}) \ .$$

The superfields $W_\alpha, \bar{W}_{\dot{\alpha}}$ satisfy the Bianchi's identities $\nabla^\alpha \star W_\alpha + \bar{\nabla}^{\dot{\alpha}} \star \bar{W}_{\dot{\alpha}} = 0$. We pay attention that the superstrengths W_α and $\bar{W}_{\dot{\alpha}}$ include the parameter of nonanticommutativity $\mathcal{C}^{\alpha\beta}$ by definition.

Classic action for $\mathcal{N} = 1/2$ SYM theory is written using superfield strengths W_α and $\bar{W}_{\dot{\alpha}}$ as follows

$$S = \frac{1}{2g^2} \int d^6 z \, \mathrm{tr} W^\alpha W_\alpha + \frac{1}{2g^2} \int d^6 \bar{z} \, \mathrm{tr} \bar{W}^{\dot{\alpha}} \bar{W}_{\dot{\alpha}} \ . \tag{10}$$

One can check that the action (10) can be written in the form [2]

$$\int d^2\theta \, \mathrm{tr}\, W^2 = \int d^2\theta \, \mathrm{tr} W^2|_{\mathcal{C}=0} + \int d^4x \, \mathrm{tr}(-i\mathcal{C}^{\alpha\beta} f_{\alpha\beta} \bar{\lambda}^2 - \frac{1}{2}\mathcal{C}^2 (\bar{\lambda}^2)^2) \ . \tag{11}$$

Here $W|_{\mathcal{C}=0}$ is the superfield strength of the conventional SYM theory.

Variation of the classic action

$$\delta S = \frac{i}{g^2}\mathrm{tr}\int d^8z\ \left[(\mathrm{e}_\star^{-V}\star D^\alpha \mathrm{e}_\star^V),(\mathrm{e}_\star^{-V}\star\delta\mathrm{e}_\star^V)\right]\star W_\alpha = -\frac{i}{g^2}\int d^8z\,(\Delta V)_\star\star(\nabla^\alpha\star W_\alpha)\ ,$$

leads to classical equations of motion in superfield form

$$\nabla^\alpha\star W_\alpha = 0\ . \tag{12}$$

The dynamics of chiral scalar matter in the (anti)fundamental representation of the gauge group minimally coupled to the gauge field is described by the action

$$S=\int d^8z\,\bar\Phi_f\star\mathrm{e}_\star^V\star\Phi_f+\int d^8z\,\tilde\Phi_{\bar f}\star\mathrm{e}_\star^{-V}\star\bar{\tilde\Phi}_{\bar f}+\int d^6z\,\mathcal{W}_\star(\Phi_f,\tilde\Phi_{\bar f})+\int d^6\bar z\,\bar{\mathcal{W}}_\star(\bar\Phi_f,\bar{\tilde\Phi}_{\bar f})\ , \tag{13}$$

where $\Phi,\tilde\Phi$ are chiral superfields and $\bar\Phi,\bar{\tilde\Phi}$ antichiral superfields. The kinetic action for the adjoint representation of the matter fields is

$$S=\int d^8z\,\mathrm{Tr}(\mathrm{e}_\star^{-V}\star\bar\Phi\star\mathrm{e}_\star^V\star\Phi)\ . \tag{14}$$

The component field redefinition analogous to the Seiberg-Witten map in noncommutative non-supersymmetric theories, such that these fields transform canonically under the gauge transformation was found in [2].

In order to formulate the superfield background field method [8], [9] for NAC SYM theories we have to perform the background-quantum splitting $\mathrm{e}_\star^V\to\mathrm{e}_\star^\Omega\star e_\star^v$ (or $\mathrm{e}_\star^V\to\mathrm{e}_\star^v\star\mathrm{e}_\star^{\bar\Omega}$) where $\Omega,(\bar\Omega)$, v are the background and quantum superpotentials respectively. Then we have to write the covariant derivatives in gauge-(anti)chiral representation as $\nabla_\alpha=\mathrm{e}_\star^{-v}\star\nabla_\alpha\star\mathrm{e}_\star^v, \bar\nabla_{\dot\alpha}=\bar D_{\dot\alpha}$ ($\nabla_\alpha=D_\alpha,\bar\nabla_{\dot\alpha}=\mathrm{e}_\star^v\star\bar\nabla_{\dot\alpha}\star\mathrm{e}_\star^{-v}$) with the standard transformation rules in respect to two gauge transformations types (quantum and background). The covariantly (anti)chiral superfields $\nabla_\alpha(\mathrm{e}_\star^{-\Omega}\star\bar\Phi)=\bar\nabla_{\dot\alpha}(\mathrm{e}_\star^{\bar\Omega}\star\Phi)=0$ are splitted linearly into background and a quantum parts. Background field quantization consists in use of gauge fixing which explicitly breaks the quantum gauge invariance while preserves manifest background gauge invariance. The procedure in NAC case is analogous to conventional one [8], [9] and means a replacement the point-products of superfields with the star-products.

In next sections we consider the effective action induced by the quantum matter and gauge fields on a special background of $U(1)$ vector multiplet superfield.

4 The gauge invariant effective action induced by the matter in the fundamental representation

We consider the theory with action (13) where the superfields $\tilde\Phi$ are absent. For one-loop calculations we have to find a quadratic over quantum fields part of the classical action. After background-quantum splitting defined earlier one can obtain

$$S_{(2)}=\frac{1}{2}\int d^8z\,(\bar\Phi_c^T,\Phi_c^T)\star\hat H_\star\star\begin{pmatrix}\Phi_c\\ \bar\Phi_c\end{pmatrix},\quad \hat H_\star=\begin{pmatrix}\nabla^2\star\bar\nabla^2 & \bar m\nabla^2\\ m\bar\nabla^2 & \bar\nabla^2\star\nabla^2\end{pmatrix}\ , \tag{15}$$

where the 'masses' are $m=\mathcal{W}''_{\Phi\Phi}(\Phi),\bar m=\bar{\mathcal{W}}''_{\bar\Phi\bar\Phi}(\bar\Phi)$.

The one-loop correction to the effective action is formally given by the expression

$$i\Gamma^{(1)} = -\ln \mathrm{Det}\hat{H}_\star = -\mathrm{Tr}\ln \hat{H}_\star = \zeta'(0)\ , \tag{16}$$

where $\zeta'(0|H) = \zeta'(\epsilon|H)|_{\epsilon=0}$ and the zeta-function is defined as follows

$$\zeta(\epsilon|H) = \frac{1}{\Gamma(\epsilon)}\int_0^\infty ds s^{\epsilon-1}\,\mathrm{Tr}(\mathrm{e}_\star^{s\hat{H}_\star}). \tag{17}$$

Here Tr is the superspace functional trace $\mathrm{Tr}A = \int d^8z A(z,z')\delta^8(z-z')|_{z'=z}$ and $\mathrm{e}_\star^{s\hat{H}_\star} = 1 + s\hat{H}_\star + \frac{s^2}{2}\hat{H}_\star \star \hat{H}_\star + ...$ Further we follows the procedure proposed in [11]. Evaluating the effective action on the base of proper-time techniques consists in two steps: calculation of the heat kernel, finding the trace and then it renormalization.

First of all we obtain the useful representation of the ζ-function (17). Separating the diagonal and non-diagonal parts of the operator $\hat{H}_\star$ in (17) we rewrite the ζ-function as

$$\zeta(\epsilon|H) = \frac{1}{\Gamma(\epsilon)}\int_0^\infty ds\cdot s^{\epsilon-1}\ \mathrm{Tr}\left(\mathrm{e}_\star^{s\begin{pmatrix} 0 & \bar{m}\nabla^2 \\ m\bar{\nabla}^2 & 0\end{pmatrix}}\mathrm{e}_\star^{s\begin{pmatrix} \nabla^2\star\bar{\nabla}^2 & 0 \\ 0 & \bar{\nabla}^2\star\nabla^2\end{pmatrix}}\right) \tag{18}$$

$$= \int d^8z \int_0^\infty \frac{ds}{\Gamma(\epsilon)} s^{\epsilon-1}\sum_{n=0}^\infty \frac{s^{2n}}{(2n)!}(m\bar{m})^n \frac{d^n}{ds^n}\mathrm{e}_\star^{s\nabla^2\star\bar{\nabla}^2}\delta^8(z-z')|_{z'=z} + (\nabla^2 \leftrightarrow \bar{\nabla}^2)$$

$$= \int d^6z \int_0^\infty \frac{ds}{\Gamma(\epsilon)} s^{\epsilon-1}\sum_{n=0}^\infty \frac{s^{2n}}{(2n)!}(m\bar{m})^n \frac{d^n}{ds^n}\mathrm{e}_\star^{s\bar{\nabla}^2\star\nabla^2}\star\bar{\nabla}^2\delta^8(z-z')|_{z'=z} + \int d^6\bar{z}(\nabla^2 \leftrightarrow \bar{\nabla}^2)\ .$$

Here we used property $d^8z = d^6z\bar{\nabla}^2 = d^6\bar{z}\nabla^2$ and fulfilled integration by parts. Also we suggest that the 'masses' are slowly varying. The 'mass' dependence in (18) is accompanied by derivatives of the (anti)chiral kernels. It is convenient to separate these derivatives from 'masses'. It can be done by repeated integration by parts. Direct calculation with keeping in mind a limit $\epsilon \to 0$ leads to the following representation of ζ-function

$$\zeta(\epsilon|H) = \frac{1}{\Gamma(\epsilon)}\int_0^\infty ds\cdot s^{\epsilon-1}\mathrm{e}^{-m\bar{m}s}\left(K_+(s) + K_-(s)\right)\ , \tag{19}$$

with the chiral and antichiral heat kernel traces defined as

$$K_+(s) = \frac{1}{2}\int d^6z\mathrm{e}_\star^{s\Box_+}\star\bar{\nabla}^2\delta^8(z-z')|_{z=z'}\ , K_-(s) = \frac{1}{2}\int d^6\bar{z}\,\mathrm{e}_\star^{s\Box_-}\star\nabla^2\delta^8(z-z')|_{z=z'}\ . \tag{20}$$

In above expressions we used the Laplace-type operators acting in the space of covariantly (anti)chiral superfields ($\Box_+\Phi = \bar{\nabla}^2\star\nabla^2\star\Phi$, $\Box_-\bar{\Phi} = \nabla^2\star\bar{\nabla}^2\star\bar{\Phi}$)

$$\Box_+ = \Box_\star - iW^\alpha\star\nabla_\alpha - \frac{i}{2}(\nabla^\alpha\star W_\alpha), \quad \Box_- = \Box_\star - i\bar{W}^{\dot{\alpha}}\star\bar{\nabla}_{\dot{\alpha}} - \frac{i}{2}(\bar{\nabla}^{\dot{\alpha}}\star\bar{W}_{\dot{\alpha}})\ , \tag{21}$$

where $\Box_\star = \frac{1}{2}\nabla^{\alpha\dot{\alpha}}\star\nabla_{\alpha\dot{\alpha}}$. Relations (16, 19, 20) define the gauge invariant one-loop effective action for the theory under consideration.

Further we will consider an approximation of a covariantly constant on-shell background vector multiplet where the of effective action can be carried up the very end and a result can be expressed in a closed form. Such a background is defined as follows

$$\nabla_{\alpha\dot{\alpha}}\star W_\beta = 0, \quad \nabla^\alpha\star W_\alpha = 0. \tag{22}$$

The one-loop contribution to the effective action is found using the methods formulated in the Refs [12], [11] and taking into account the property (5) of the star-product. As a result one gets

$$\begin{aligned}\Gamma^{(1)} &= -\frac{1}{(4\pi)^2}\int d^6z\, W^2 \ln\frac{m}{\Lambda} + c.c. \\ &+\frac{1}{(4\pi)^2}\int_0^\infty ds\cdot s\, \mathrm{e}^{-m\bar{m}s}\int d^8z\, W^2\star\bar{W}^2\star\zeta_\star(s\mathcal{N}, s\bar{\mathcal{N}})\end{aligned} \tag{23}$$

where $\mathcal{N}_\alpha^\beta = D_\alpha W^{\beta 3}$, $\bar{\mathcal{N}}_{\dot\alpha}^{\dot\beta} = \bar{D}_{\dot\alpha}\bar{W}^{\dot\beta}$ and function $\zeta(x,y)$ has been introduced in [12]

$$\zeta_\star(x,y) = \frac{y^2\star(\cos_\star x - 1) - x^2\star(\cos_\star y - 1)}{x^2\star y^2\star(\cos_\star x - \cos_\star y)}\ . \tag{24}$$

Thus, in the theory under consideration with the operators $\Box_\pm$ including only left star-products, the heat trace expansion and hence the effective action is completely defined by ones in conventional superfield theory, so the only difference is presence of $\star$-products in final results instead of ordinary products.

5 Heat kernel and the effective action in the NAC SYM theory

In this section we consider the one-loop contributions to the effective action of the gauge fields and ghosts. We study a theory with $SU(2)$ gauge group broken to $U(1)$ and assume that the background is described by on-shell Abelian superfield (22).

5.1 Features of the background-quantum splitting in the NAC SYM theory

We describe a structure of the background-quantum splitting in the NAC SYM model and point out a role of the operators $T^\star_{c,s}$ (6) for the fields in an adjoint representation.

Using the background-quantum splitting $\nabla \to \mathrm{e}^{-v}\nabla\mathrm{e}^{v}$ and the standard gauge fixing function $\chi = \nabla^2 v$ leads to the following quadratic part of $\mathcal{N} = 1/2$ SYM action with $SU(2)$ gauge group for the quantum superfield v (see [8] for some details in conventional superfield theory)

$$S^{(2)}_{gauge+gf} = -\frac{1}{2g^2}\mathrm{tr}\int d^8z\, v\star H^{(v)}_\star\star v \tag{25}$$

where the operator $H^v_\star$ has the form

$$H^{(v)}_\star = \Box_\star - i\{W^\alpha, \star\nabla_\alpha\} - i\{\bar{W}^{\dot\alpha}, \star\bar{\nabla}_{\dot\alpha}\} \tag{26}$$

Similarly, the quadratic part of the ghost action has the form

$$S^{(2)}_{ghosts} = \mathrm{tr}\int d^8z\,(\bar{c}'c - c'\bar{c} + \bar{b}b)\ . \tag{27}$$

[3] $\mathcal{N}_\alpha^\beta\mathcal{N}_\beta^\delta = \delta_\alpha^\delta D^2W^2 = \delta_\alpha^\delta\mathcal{N}^2$

where all ghost superfields are background covariantly (anti)chiral. The quantum superfield v and the ghost superfields c, c', b belong to Lie algebra $su(2)$. It means that $v = v^a\tau_a$ and the same true for the ghost superfields. Here $\tau_a = \frac{1}{\sqrt{2}}\sigma_a$ are the generators of $su(2)$ algebra satisfying the relations $[\tau_a, \tau_b] = i\sqrt{2}\epsilon_{abc}\tau_c$, $\mathrm{tr}(\tau_a\tau_b) = \delta_{ab}$.

Structure of the operator $H^{(v)}_\star$ (26) for the background belonging to the Abelian subgroup $U(1)$ (in this case $W = W^3\tau_3$) can be simplified because of the property of the star-operator T^*_c. The operator $H^{(v)}_\star$ (26) includes the term $W^\alpha \star \nabla_\alpha$. We write $\nabla_\alpha = D_\alpha - i\Gamma_\alpha$ and consider the term in $H^{(v)}_\star$ containing only D_α

$$\begin{aligned} v\{W, \star Dv\} &= \tau^a\tau^c\tau^b(v^aW^c \star Dv^b) + \tau^a\tau^b\tau^c(v^aDv^b \star W^c) \\ &= \frac{1}{2}\tau^a[\tau^c, \tau^b]v^a[W^c, \star Dv^b] + \frac{1}{2}\tau^a\{\tau^c, \tau^b\}v^a\{W^c, \star Dv^b\}\ . \end{aligned} \tag{28}$$

As a result one gets

$$\mathrm{tr}\left(v\{W^\alpha \star D_\alpha v\}\right) = \mathrm{tr}\left(\tau^a[\tau^c, \tau^b]\right) v^aW^\alpha_c T^\star_c D_\alpha v^b + \mathrm{tr}\left(\tau^a\{\tau^c, \tau^b\}\right) v^aW^\alpha_c T^\star_s D_\alpha v^b\ , \tag{29}$$

where we have used the definitions $T^\star_{c(s)} = \frac{1}{2}(\star \pm \star^{-1})$ to rewrite the $[W, \star Dv]$ and $\{W, \star Dv\}$. The last term in (29) is equal to zero because of $\mathrm{tr}(\tau^a\{\tau^c, \tau^b\}) = 0$, while in the first term only component W_3 survives. After redefinition of the gauge field components v^1 and v^2 as follows $\chi = \frac{1}{\sqrt{2}}(v^1 + iv^2)$, $\quad \tilde{\chi} = \frac{1}{\sqrt{2}}(v^1 - iv^2)$ the first term can be rewritten in the form

$$\sqrt{2}\chi W^\alpha_3 T^\star_c D_\alpha\tilde{\chi} - \sqrt{2}\tilde{\chi}W^\alpha_3 T^\star_c D_\alpha\chi\ , \tag{30}$$

while the v^3 component of the quantum superfield $v = v^a\tau_a$ do not interact with the background and totally decouple. Above we have analyzed a contribution of the operator D_α from operator ∇_α in (26). Now let us consider a contribution of another term Γ_α which forms together with D_α the supercovariant derivative $\nabla_\alpha = D_\alpha - i\Gamma_\alpha$. Its contribution to $\mathrm{tr}(v \star H^{(v)}_\star \star v)$ is given by

$$\begin{aligned} &\mathrm{tr}\left(v \star \{W \star [\Gamma, \star v]\}\right) = \\ &\mathrm{tr}\left(\frac{1}{4}[\tau_a, \tau_c]\{\tau_d, \tau_b\}\right) v^a[W^c, \star[\Gamma^d, \star v^b]] + \mathrm{tr}\left(\frac{1}{2}\{\tau_a, \tau_c\}\frac{1}{2}\{\tau_d, \tau_b\}\right) v^a\{W^c, \star[\Gamma^d, \star v^b]\} \\ &+\mathrm{tr}\left(\frac{1}{2}\{\tau_a, \tau_c\}\frac{1}{2}[\tau_d, \tau_b]\right) v^a\{W^c, \star\{\Gamma^d, \star v^b\}\} + \mathrm{tr}\left(\frac{1}{2}[\tau_a, \tau_c]\frac{1}{2}[\tau_d, \tau_b]\right) v^a[W^c, \star\{\Gamma^d, \star v^b\}]\ . \end{aligned} \tag{31}$$

It is easy to see that the first and the third terms are equal to zero due to the trace properties of τ-matrices. The second term is proportional to $v^3W^\alpha T^\star_s\Gamma_\alpha T^\star_s v^3$ and will not give a contribution to the effective action, because of the property (7) for star-operator $T^\star_s$. The last term together with (30) can be rewritten as $\chi[-iW^\alpha_3 T^\star_c\nabla_\alpha]\tilde{\chi}$, where now $\nabla_\alpha = D_\alpha - i\Gamma_\alpha T^\star_c$. Appearance of $T^\star_c$ was stipulated by the adjoint representation.

As a result we found that the contributions of v^3- component of the quantum gauge multiplet totally decouple. Moreover, according to the property of $T^\star_s$-product, given in the Section 2, we see that theirs contributions the one-loop effective action are absent. Non-trivial contribution to the effective action is generated by the components v^1 and v^2 or by their linear combinations χ and $\tilde{\chi}$. We want to emphasize that the action for the $\chi, \tilde{\chi}$ corresponds to non-Abelian superfield model where the star-operator $T^\star_c$ plays the role of an internal symmetry generator including whole star-structure of the initial theory. Further we study a construction of the heat kernel and the effective action of the theory under consideration generalizing the techniques [10] for NAC superspace.

5.2 The heat kernels on the covariantly constant background

Above we have shown that the second variational derivative of the action in sector of the superfields $\chi, \tilde{\chi}$ has the form $S^{(2)}_{gauge+FG} = \int d^8z\, \chi H^{\chi}_{\tilde{\star}} \tilde{\chi}$ where the operator $H^{\chi}_{\tilde{\star}}$ defined as

$$H^{\chi}_{\tilde{\star}} = \Box_{\tilde{\star}} - iW^{\alpha}_{\tilde{\star}} \nabla_{\tilde{\star}\,\alpha} - i\bar{W}^{\dot{\alpha}}_{\tilde{\star}} \bar{\nabla}_{\tilde{\star}\,\dot{\alpha}} \; , \tag{32}$$

and the notations $W_{\tilde{\star}} = WT^{\star}_c$ and $\nabla_{\tilde{\star}}$ for $D - i\Gamma T^{\star}_c$ were used. We define the Green function $G(z,z')$ of the operator $H^{\chi}_{\tilde{\star}}$ by the equation $H^{\chi}_{\tilde{\star}} G(z,z') = -\delta^8(z-z')$. Then one introduces the heat kernel $K_{\chi}(z,z'|s)$ associated with this Green function as $G(z,z') = \int_0^{\infty} ds\, K(z,z'|s)\, \mathrm{e}^{-\varepsilon s}|_{\varepsilon\to+0}$. It means that formally $K_{\chi}(z,z'|s) = \mathrm{e}^{sH^{\chi}_{\tilde{\star}}} \delta^8(z-z')$. The one-loop contribution of the gauge superfields to the effective action is proportional to $\mathrm{Tr}(K_{\chi})$ and gauge invariant due to the gauge transformation law

$$K_{\chi}(z,z') \to \mathrm{e}^{i\Lambda(z)}_{\tilde{\star}} K_{\chi}(z,z'|s) \mathrm{e}^{-i\Lambda(z')}_{\tilde{\star}}.$$

We rewrite the kernel K_{χ} in the form

$$K_{\chi}(z,z'|s) = \mathrm{e}^{s(\Box_{\tilde{\star}} - iW^{\alpha}_{\tilde{\star}} \nabla_{\tilde{\star}\,\alpha} - i\bar{W}^{\dot{\alpha}}_{\tilde{\star}} \bar{\nabla}_{\tilde{\star}\,\dot{\alpha}})} \times (\delta^8(z-z') I(z,z')) \; , \tag{33}$$

where bi-scalar $I(z,z')$ satisfies the equation $\zeta_A(z,z') \nabla^A_{\tilde{\star}} I(z,z') = 0$ and boundary condition $I(z,z) = 1$. Further we will use the techniques developed in [10] adopting it to the NAC superspace.

In order to calculate (33) we, first of all, write the operator $H^{\chi}_{\tilde{\star}}$ (32) as follows $H^{\chi}_{\tilde{\star}} = \Box_{\tilde{\star}} + V$ where $V = -iW^{\alpha}_{\tilde{\star}} \nabla_{\tilde{\star}\,\alpha} - i\bar{W}^{\dot{\alpha}}_{\tilde{\star}} \bar{\nabla}_{\tilde{\star}\,\dot{\alpha}}$ and decompose the operator $\mathrm{e}^{sH^{\chi}_{\tilde{\star}}}$ as

$$\mathrm{e}^{s(\Box_{\tilde{\star}}+V)} = \cdots \times \mathrm{e}^{+\frac{s^3}{2}[V,[\Box_{\tilde{\star}},V]] + \frac{s^3}{6}[\Box_{\tilde{\star}},[\Box_{\tilde{\star}},V]]} \mathrm{e}^{\frac{s^2}{2}[\Box_{\tilde{\star}},V]} \mathrm{e}^{sV} \mathrm{e}^{s\Box_{\tilde{\star}}} \; . \tag{34}$$

It follows from (9) that for $\nabla_{\tilde{\star}\,\alpha\dot{\alpha}} W_{\tilde{\star}\,\beta} = 0$ the first commutator becomes

$$[\Box_{\tilde{\star}}, V] = (iW_{\alpha} T^{\star}_c \bar{W}_{\dot{\alpha}} + i\bar{W}_{\dot{\alpha}} T^{\star}_c W_{\alpha}) T^{\star}_c \nabla^{\alpha\dot{\alpha}}_{\tilde{\star}} = 0 \; , \tag{35}$$

because of the property $W_{\alpha}(\star + \star^{-1})\bar{W}_{\dot{\alpha}} = -\bar{W}_{\dot{\alpha}}(\star^{-1} + \star)W_{\alpha}$ which is valid for the chosen Abelian background. This identity leads to convenient factorization of the kernel in the form (see [10] for details in conventional superfield theory)

$$K_{\chi}(z,z'|s) = U_{\tilde{\star}}(s)\, \tilde{K}(\zeta|s)\, \zeta^2 \bar{\zeta}^2\, I(z,z'), \quad U_{\tilde{\star}}(s) = \mathrm{e}^{-is(W^{\alpha}_{\tilde{\star}} \nabla_{\tilde{\star}\,\alpha} + \bar{W}^{\dot{\alpha}}_{\tilde{\star}} \bar{\nabla}_{\tilde{\star}\,\dot{\alpha}})} \; . \tag{36}$$

Here we have used the chiral basis with coordinates $(y^{\alpha\dot{\alpha}}, \theta^{\alpha}, \theta^{\dot{\alpha}})$ for calculations and the presentation $\delta^8(z-z') = \delta^4(\zeta^{\alpha\dot{\alpha}})\zeta^2\bar{\zeta}^2$. The translation invariant interval components $\zeta^A(z,z')$ in the chiral basis are defined by

$$\zeta^A = (\zeta^{\alpha\dot{\alpha}}, \zeta^{\alpha}, \bar{\zeta}^{\dot{\alpha}}) = ((y-y')^{\alpha\dot{\alpha}} - i(\theta-\theta')^{\alpha}\bar{\theta}'^{\dot{\alpha}}, (\theta-\theta')^{\alpha}, (\bar{\theta}-\bar{\theta})^{\dot{\alpha}}) \; .$$

The Schwinger type heat kernel $\tilde{K}(\zeta|s)$ in (36) can be calculated by the various methods and it is well known

$$\tilde{K}(\zeta|s) = \frac{i}{(4\pi s)^2} \exp(-\frac{1}{2} \mathrm{tr} \ln_{\tilde{\star}} \frac{\sin_{\tilde{\star}} sF/2}{sF/2}) \mathrm{e}^{-\frac{1}{2s}\zeta(\frac{sF}{2}\cot_{\tilde{\star}}\frac{sF}{2})\zeta} \; , \tag{37}$$

where $F^{\beta\dot\beta}_{\alpha\dot\alpha} = \delta^{\dot\beta}_{\dot\alpha} f^{\beta}_{\alpha} + \delta^{\beta}_{\alpha} \bar f^{\dot\beta}_{\dot\alpha}$ and $f^{\beta}_{\alpha}, \bar f^{\dot\beta}_{\dot\alpha}$ are the spinor components of the Abelian strengths F_{mn}[4]. The contraction over indices in the right exponent goes only for $\zeta^{\alpha\dot\alpha}$ components. The functions $f_{\tilde\star}(x)$ are given as an expansion $f_{\tilde\star}(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}}{n!}(xT^{\star}_c(xT^{\star}_c(...T^{\star}_c x)))$.

Next step of calculations is obtaining the action of the operator $U_{\tilde\star}(s)$ in (36). This operator contains covariant derivatives which act on the interval components $U_{\tilde\star}(s)\zeta^A = \zeta^A(s)U_{\tilde\star}(s)$. Hence we should consider the adjoint action of U on ζ^A. Introducing the notation $\mathcal{N}^{\beta}_{\tilde\star\alpha} = \nabla_{\tilde\star\alpha} W^{\beta}_{\tilde\star}$, $\bar{\mathcal{N}}^{\dot\beta}_{\tilde\star\dot\alpha} = \bar\nabla_{\tilde\star\dot\alpha} \bar W^{\dot\beta}_{\tilde\star}$ we have for adjoint action U on the interval components

$$\zeta^{\alpha}(s) = \zeta^{\alpha} + W^{\delta}_{\tilde\star}\left(\frac{\mathrm{e}^{-is\mathcal{N}_{\tilde\star}} - 1}{\mathcal{N}_{\tilde\star}}\right)^{\alpha}_{\delta}, \quad \bar\zeta^{\dot\alpha}(s) = \bar\zeta^{\dot\alpha} + \bar W^{\dot\delta}_{\tilde\star}\left(\frac{\mathrm{e}^{-is\bar{\mathcal{N}}_{\tilde\star}} - 1}{\bar{\mathcal{N}}_{\tilde\star}}\right)^{\dot\alpha}_{\dot\delta}, \tag{38}$$

$$\zeta^{\alpha\dot\alpha}(s) = \zeta^{\alpha\dot\alpha} + \int_0^s d\tau W^{\alpha}_{\tilde\star}(\tau)\bar\zeta^{\dot\alpha}(\tau)\ , \tag{39}$$

where $W^{\alpha}(s)_{\tilde\star} = W^{\beta}_{\tilde\star}(e^{-is\mathcal{N}_{\tilde\star}})^{\alpha}_{\beta}$.

Next step is calculation of $U_{\tilde\star} I(z,z')$ in (36). To do that we write a differential equation

$$i\frac{d}{ds}U_{\tilde\star}(s)I(z,z') = U_{\tilde\star}(s)(W_{\tilde\star}\nabla_{\tilde\star} + \bar W_{\tilde\star}\bar\nabla_{\tilde\star})U^{-1}_{\tilde\star}(s)U_{\tilde\star}(s)I(z,z') \tag{40}$$

and solve it. Thus, we should construct the operators $\nabla_{\tilde\star A}(s)$ and act on $I(z,z')$. We pay attention that the procedure of calculations, we discuss here, preserves manifest gauge invariance. Therefore, to simplify the calculations, we can impose any appropriate gauge on background superfield. The treatment with $I(z,z')$ are very much simplified under conditions $I(z,z') = 1$ which is equivalent to the Fock-Schwinger gauge $\zeta^A \star \Gamma_{\tilde\star A} = 0$ or $\nabla_{\tilde\star A} I(z,z') = -iI(z,z')\Gamma_{\tilde\star A}$ (see details in [10] for conventional superspace theories).

If in the chiral basis we have a supercovariant derivative $\nabla'_{\tilde\star A}$ in the point z', then the supercovariant derivative $\nabla_{\tilde\star}$ satisfying the Fock-Schwinger gauge in point z has the form

$$\nabla_{\tilde\star A} = \mathrm{e}^{+iy'^{m}Q_m + i\theta' Q}\mathrm{e}^{+i\bar\theta'\bar Q}\mathrm{e}^{+y^{\alpha\dot\alpha}\nabla'_{\tilde\star\alpha\dot\alpha} + \theta^{\alpha}\nabla'_{\tilde\star\alpha}}\mathrm{e}^{+\bar\theta^{\dot\beta}\bar\nabla'_{\tilde\star\dot\beta}}\mathrm{e}^{-\bar\theta'^{\dot\alpha}\bar D_{\dot\alpha}}\mathrm{e}^{-y'^{\beta\dot\beta}\partial_{\beta\dot\beta} - \theta'^{\alpha}D_{\alpha}} \times \tag{41}$$
$$(\nabla'_{\tilde\star A}) \times \quad \mathrm{e}^{y'^{\beta\dot\beta}\partial_{\beta\dot\beta} + \theta'^{\alpha}D_{\alpha}}\mathrm{e}^{\bar\theta'^{\dot\alpha}\bar D_{\dot\alpha}}\mathrm{e}^{-\bar\theta^{\dot\beta}\bar\nabla'_{\tilde\star\dot\beta}}\mathrm{e}^{-y^{\alpha\dot\alpha}\nabla'_{\tilde\star\alpha\dot\alpha} - \theta^{\alpha}\nabla'_{\tilde\star\alpha}}\mathrm{e}^{-i\bar\theta'\bar Q}\mathrm{e}^{-iy'^{m}Q_m - i\theta' Q}$$

The relation (41) leads to explicit expressions for the connections in the Fock-Schwinger gauge:

$$\bar\nabla_{\tilde\star\dot\beta} - \bar D_{\dot\beta} = -i\Gamma_{\tilde\star\dot\beta} = 0\ , \tag{42}$$

$$\nabla_{\tilde\star\beta} - D_{\beta} = -i\Gamma_{\tilde\star\beta} = \frac{1}{2}\zeta^{\dot\beta}_{\beta L}\bar W'_{\tilde\star\dot\beta} - \frac{i}{2}\zeta_{\beta}(\bar\zeta^{\dot\beta}\bar W'_{\tilde\star\dot\beta}) - \frac{1}{2}\zeta^{\alpha\dot\alpha}_{L}\bar\zeta^{\dot\beta}F'_{\tilde\star\beta\dot\beta,\alpha\dot\alpha} + i\bar\zeta^2(W'_{\tilde\star\beta} + \zeta^{\alpha}(\nabla'_{\alpha}W'_{\tilde\star\beta}))\ ,$$

$$\nabla_{\tilde\star\beta\dot\beta} - \partial_{\beta\dot\beta} = -i\Gamma_{\tilde\star\beta\dot\beta} = -\frac{i}{2}\zeta^{\alpha\dot\alpha}_{L}F'_{\tilde\star\alpha\dot\alpha,\beta\dot\beta} + \bar\zeta_{\dot\beta}W'_{\tilde\star\beta} + \frac{1}{2}\zeta_{\beta}\bar W'_{\tilde\star\dot\beta} + \bar\zeta_{\dot\beta}\zeta^{\alpha}(\nabla'_{\tilde\star\alpha}W'_{\tilde\star\beta}).$$

First of these relations is the consequence of the supercovariant derivative forms in the chiral basis (8). Using (42) one can find the solution of the equation (40) in the form

$$U_{\tilde\star}(s)I(z,z') = \exp_{\tilde\star}\left(i\int_0^s d\tau(\frac{1}{2}W_{\tilde\star\beta}(\tau)\zeta^{\beta\dot\beta}(\tau)\bar W'_{\tilde\star\dot\beta} + \frac{i}{2}W^{\beta}_{\tilde\star}(\tau)\zeta_{\beta}(\tau)\bar\zeta^{\dot\beta}(\tau)\bar W'_{\tilde\star\dot\beta}\right. \tag{43}$$

[4]Also we point out the useful equation $\nabla_{\tilde\star\alpha\dot\alpha}\tilde K + \left(\frac{iF}{\mathrm{e}^{isF}_{\tilde\star} - 1}\right)^{\beta\dot\beta}_{\alpha\dot\alpha}\zeta_{\beta\dot\beta}\tilde\star\tilde K = 0$, which allows us to get any order derivatives of the Schwinger type kernel.

$$+\frac{1}{2}W^{\beta}_{\tilde\star}(\tau)\zeta^{\alpha\dot\alpha}(\tau)\bar\zeta^{\dot\beta}(\tau)F'_{\tilde\star\,\beta\dot\beta\alpha\dot\alpha} - i\bar\zeta^2(\tau)W^{\beta}_{\tilde\star}(\tau)W'_{\tilde\star\beta} - i\bar\zeta^2(\tau)W^{\beta}_{\tilde\star}(\tau)\zeta^{\alpha}(\tau)f'_{\tilde\star\,\alpha\beta)}\Big) \ .$$

Substituting the $\zeta^A(s)$ (38) and $U_{\tilde\star}I(z,z')$ (43) into (36) and taking into account (37) one gets finally the kernel

$$K_\chi(z,z'|s) = \tilde K(\zeta|s)\,\zeta^2(s)\bar\zeta^2(s)U_{\tilde\star}(s)I(z,z')\ . \tag{44}$$

determining the effective action.

Now we discuss a structure of kernels corresponding to the ghost or to any adjoint chiral matter contribution to the effective action. First of all we point out that the following relations take place in on-shell Abelian background

$$\bar\nabla^2_{\tilde\star}\mathrm{e}^{sH^\chi_{\tilde\star}} = \bar\nabla^2_{\tilde\star}\mathrm{e}^{s\nabla^2_{\tilde\star}\bar\nabla^2_{\tilde\star}} = \mathrm{e}^{s\bar\nabla^2_{\tilde\star}\nabla^2_{\tilde\star}}\bar\nabla^2_{\tilde\star} = \mathrm{e}^{s\Box_{\tilde\star +}}\bar\nabla^2_{\tilde\star}, \tag{45}$$

$$\nabla^2_{\tilde\star}\mathrm{e}^{sH^\chi_{\tilde\star}} = \nabla^2_{\tilde\star}\mathrm{e}^{s\bar\nabla^2_{\tilde\star}\nabla^2_{\tilde\star}} = \mathrm{e}^{s\nabla^2_{\tilde\star}\bar\nabla^2_{\tilde\star}}\nabla^2_{\tilde\star} = \mathrm{e}^{s\Box_{\tilde\star -}}\nabla^2_{\tilde\star}\ . \tag{46}$$

Let us introduce the chiral and antichiral heat kernels

$$K_+(z,z'|s) = \bar\nabla^2_{\tilde\star}K_\chi(z,z'|s) = \bar\nabla'^2_{\tilde\star}K_\chi(z,z'|s), \tag{47}$$
$$K_-(z,z'|s) = \nabla^2_{\tilde\star}K_\chi(z,z'|s) = \nabla'^2_{\tilde\star}K_\chi(z,z'|s)\ .$$

The functions $G_\pm(z,z')$ defined as $G_\pm(z,z') = \int_0^\infty ds\, K_\pm(z,z'|s)\,\mathrm{e}^{-\varepsilon s}|_{\varepsilon\to+0}$ satisfy the equations $\Box_{\tilde\star\pm}G_\pm(z,z') = -\delta_\pm(z,z')$ where $\Box_{\tilde\star\pm}$ are the (anti)chiral d'Alambertians depending on background superfield in NAC superspace. It means that the functions (47) are the kernels associated with the operators $\Box_{\tilde\star\pm}$. Namely these kernels determine the one-loop contribution to the effective action from any chiral matter in the adjoint representation. Since the kernels (47) are from the kernel K_χ we can substitute (44) into (47) and find these kernels

$$K_+(z,z'|s) = \bar\nabla'^2_{\tilde\star}K_\chi = -\tilde K(\zeta|s)\zeta^2(s)U_{\tilde\star}(s)\,I(z,z') \tag{48}$$

$$K_-(z,z'|s) = \nabla'^2_{\tilde\star}K_\chi = -\tilde K(\zeta|s)\,\bar\zeta^2(s)\mathrm{e}_{\tilde\star}^{-\frac{1}{2}\zeta^\alpha(s)\zeta_{\alpha\dot\alpha}(s)\bar W^{\dot\alpha}_{\tilde\star}(s)}U_{\tilde\star}(s)\,I(z,z')\ . \tag{49}$$

These relations determine a contribution of the chiral adjoint matter including ghost superfields into the one-loop effective action.

In next subsection we consider heat traces associated with the heat kernels (44, 48, 49) and, particularly, coefficient a_2 in the Schwinger-De Witt expansion of the low-energy effective action.

5.3 Gauge fields and ghosts contribution to the effective action $SU(2)$ SYM theory

The one-loop contribution $\Gamma^{(1)}$ to the effective action for NAC SYM theory is defined with the help of the ζ-functions $\zeta_{\chi,\pm}(\epsilon)$ corresponding to the operators $H^\chi_\star$, $\Box_{\tilde\star\pm}$ respectively

$$\Gamma^{(1)} = \Gamma^{(1)}_\chi + \Gamma^{(1)}_{ghosts}. \tag{50}$$

Here $\Gamma^{(1)}_\chi$ is pure SYM contribution and $\Gamma^{(1)}_{ghosts}$ is ghost contribution. Each of these contributions is calculated via function $\zeta'(0)$. In its turn, the ζ-functions are given by integral representations

$$\zeta_{\chi,\pm}(\epsilon) = \frac{1}{\Gamma(\epsilon)} \int_0^\infty \frac{ds}{s^{1-\epsilon}} \mathrm{Tr} K_{\chi,\pm}(\frac{s}{\mu^2}) \tag{51}$$

where μ is the normalization point and Tr is an inherent functional trace.

General structure of $K_{\chi,\pm}(z,z'|s)$ was discussed in the Subsection 5.2 (see (44, 48, 49)). One can show that $\mathrm{Tr}K_\chi(s)$ does not contain the holomorphic contributions and, hence, the one-loop divergent contributions to the complete effective action (50) is determined exclusively by the ghosts as in the conventional case [8].

To construct the divergent part of the effective action we should consider the behavior of $K_\pm(z,z'|s)|_{z'=z}$ at small s. As usual, the kernel expansion looks like $K_\pm(z,z|s) \sim \frac{1}{s^2}(a_0(z,z) + s a_1(z,z) + s^2 a_2(z,z) + \ldots)$ The coefficient $a_2(z,z)$ is responsible for the divergences. Exact form of the kernel $K_\pm(z,z'|s)$ is given by (48, 49). The only thing we should do is to study its behavior at small s. One can show that the coefficients $a_0(z,z)=0$, $a_1(z,z)=0$. The coefficient $a_2(z,z)$ includes the products of some number of W_α, some number of superintervals ζ^A and some number of star-operators $T^\star_c$ acting on the intervals and superstrengths. Using the explicit form (6) of the operator $T^\star_c$ one can show that the final expression for $a_2(z,z)$ is a sum of one for conventional superfield theory plus a total derivative with respect to the variable θ^α which is stipulated by action of the operator $T^\star_c$ on the superinterval ζ^A. As a result one obtains

$$\int d^6z\, a_2(z,z|s) \sim \int d^6z\, W^\alpha W_\alpha \tag{52}$$

It leads to the divergent part of the effective action in the form

$$\Gamma^{(1)}_{div} = -\frac{3}{2}\cdot\frac{1}{(4\pi)^2}\int d^6z W^\alpha W_\alpha \ln\frac{\mu^2}{\Lambda^2}. \tag{53}$$

We see that the one-loop divergences on the Abelian background are analogous to classical action what provides renormalizability.

The finite parts of the effective action is analyzed by known methods (see e.g. [12])[5]. As a result one obtains

$$\Gamma^{(1)}_\chi = \frac{1}{8\pi^2}\int d^8z \int_0^\infty ds s\, \mathrm{e}^{-sm^2} W^2_\star \bar{W}^2_\star \tag{54}$$

$$\times \frac{\cosh(s\mathcal{N}_\star)-1}{(s\mathcal{N}_\star)^2}\cdot\frac{\cosh(s\bar{\mathcal{N}}_\star)-1}{(s\bar{\mathcal{N}}_\star)^2}\cdot\frac{s^2(\mathcal{N}^2_\star - \bar{\mathcal{N}}^2_\star)}{\cosh(s\mathcal{N}_\star)-\cosh(s\bar{\mathcal{N}}_\star)}\ .$$

The finite part of the chiral contributions in the effective action can be written in terms of the function $\zeta_\star$ (24):

$$\Gamma^{(1)}_{ghosts} = \frac{1}{(4\pi)^2}\int d^8z \int_0^\infty ds\, \mathrm{e}^{-sm^2} W^2_\star \bar{W}^2_\star \zeta_\star(s\mathcal{N}_\star, s\bar{\mathcal{N}}_\star) \tag{55}$$

Here m is an infrared regulator mass.

As a result, the one-loop effective action on the covariantly constant Abelian background is exactly calculated on the base of manifestly gauge invariant techniques in the NAC superspace. We emphasize a role of the star-operator $T^\star_c$ for the theories with fields in the adjoint representation.

[5] We pay attention here only on aspects associated with $\star$-structure of the theory

6 Summary

We have constructed a general procedure of calculating the effective action for SYM theory coupled to matter in $\mathcal{N} = 1/2$ nonanticommutative superspace. The model is formulated in terms of $\star$-product (3) associated with the parameter of nonanticommutativity $\mathcal{C}^{\alpha\beta}$ (1) and preserves a half of initial $\mathcal{N} = 1$ supersymmetry. The effective action is formulated in framework of superspace background field method.

We developed a proper-time techniques in $\mathcal{N} = 1$ superspace consistent with gauge invariance and $\star$-structure of the theory under consideration. Superfield heat kernel determining the structure of the one-loop effective action has been introduced for the matter in the fundamental and adjoint representations for $SU(2)$ SYM theory.

The procedure for one-loop effective action calculation has been described. We have applied this procedure to finding the low-energy effective action for the matter in the fundamental representation in an external constant Abelian vector multiplet background (22) and for the $\mathcal{N} = 1/2$ SYM model with gauge group $SU(2)$ spontaneously broken down to $U(1)$. It was shown that in case of matter in fundamental representation the low-energy effective action (23) is obtained from the corresponding effective action for the conventional $\mathcal{N} = 1$ superfield theory by inserting the $\star$-products instead of ordinary point-products. In case of SYM theory, the effective action (54), (55) is also constructed on the base of the effective action for the conventional SYM theory, where however the products are given in terms of the special star-operator $T_c^\star$ (6) introduced in the paper. We found that the models under consideration the effective action is gauge invariant and written completely in terms of $\star$-product and hence the classical $\star$-product does not get any quantum corrections.

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